Bimodal Triod-Twist Patterns

Dissertation submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfillment of the requirement for the award of the degree

Master of Science

in

Mathematics

by

Ujjal Paul
(MS23MTH020)

under the supervision of

Dr. Sourav Bhattacharya

Department of Mathematics
Visvesvaraya National Institute of Technology
Nagpur - 440010 (India)
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Declaration

I, Ujjal Paul, hereby declare that this dissertation titled "Bimodal Triod-Twist Pattern" is carried out by me at the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other institution/University.

Ujjal Paul

Date:

Certificate

This is to certify that the dissertation entitled "Bimodal Triod-Twist Pattern" is a bonafide work done by **Ujjal Paul** (Enrollment No. MS23MTH020). The project work has been completed under my guidance and supervision in a satisfactory manner. This work is a part of the requirement for the award of **Master of Science in Mathematics** from Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur. This dissertation fulfills the requirements relating the nature and standard of work for the award of Master of Science degree.

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Finally, I bow down before the almighty that has made everything possible.

Ujjal Paul

List of Notations

$\mathbb{R}$	: The set of real numbers
$\mathbb{N}$	: The set of natural numbers
$\mathbb{Z}$	: The set of integers
P	: Periodic orbit
τ	: The topological system
$f(p)$or$f(p)$	: Image of p under f
$\rho(P)$	: Rotation number of periodic orbit P
π	: The pattern
$\rho(\pi)$	: Over-rotation number of pattern π
iff	: If and only if
inf	: Infimum (or greatest lower bound)
sup	: Suprimum (or least upper bound)
$\lim p_n = p$	: Limit of sequence p_n
max	: Maximum
min	: Minimum
ϕ	: Empty set
ε	: Belongs to
$\exists$	: There exist
$f^n = fof^{n-1}$	: The n^{th} iterate of mapping f
$Per(f)$	: The set of all periodic points of f
$Fix(f)$	: The set of all fixed points of f
$\succ_s$	: 'is sharper than' symbol
$Sh(k)$	: The set of all positive integers m such that $k \succ_s m$, along with k
$P(f)$	: The set of all periods of cycles of map f

Abstract

In any branch of mathematics, one of the fundamental problems is to classify objects studied in that branch using *invariants*. Taking *invariants* as *periods*, A.N Sharkosky formulated a beautiful classification of *periodic orbits* or *cycles*, popularly called the Sharkovsky Theorem. But, lot of *cycles* have the same *period*. So, there is need to further refine this classification of *periodic orbits*. With these objective in mind, *rotation theory* for interval maps was formulated by Blokh and Misiurewicz. Later, the same authors thought of generalizing *rotation theory* even further to *triods*. A *triod T* is defined as the set of complex numbers $z \in \mathbb{C}$ such that $z^3 \in [0,1]$. The *rotation theory* for *triods* was introduced by Blokh and Misiurewicz. Later, the simplest *patterns* on *triods* with a given *rotation number*, that is, *patterns* on *triods* which don't force other *patterns* with the same *rotation number* called *triod twists* was discovered and studied by Bhattacharya. Further, the dynamics of all possible *unimodal triod-twists* (that is *triod-twist patterns* with one *critical point*) were described.

The goal of my thesis is to further carry the study in this direction and describe the dynamics of all possible *bimodal triod twist patterns* (that is *triod-twist patterns* with two *critical points*). This study has major implications in the field of *complex dynamical systems*.

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Chapter 1

Introduction

A dynamical system is a field of mathematics that focuses on how systems evolve over time. By applying concepts from real analysis and topology, we can study the complex behavior that such systems often exhibit.

In the opening chapter, titled Dynamical Systems and Sharkovskii's Theorem, we cover essential definitions and foundational ideas. We introduce key concepts such as trajectories, periods, and periodic orbits, which form the basis of understanding dynamical systems. The chapter concludes with an introduction to Sharkovskii's Theorem, which presents a fascinating method for ordering the natural numbers .

In the second chapter, Rotation Theory for Circle Maps, we start by learning the basics of rotation theory, beginning with how it works for interval maps. After that, we shift our focus to circle maps, where rotation theory becomes even more interesting. Along the way, we explore key ideas like lifting, the rotation number, and the rotation interval, and we look at how these concepts behave and what they tell us about the system. The third chapter, Over-Rotation Theory for Interval Maps, dives into several important theorems in this area. One of the main focuses is understanding how to find the over-rotation interval for continuous maps, particularly those with one or two critical points. Through this, we gain a deeper understanding of the underlying behavior and structure of these maps.

In this chapter,Introduction to Triods, we explore the concept of *triods*, a special case of *n-ods* where $n = 3$. A triod can be defined as the set of complex numbers $z \in \mathbb{C}$ such that $z^n \in [0,1]$.

Essentially, a triod consists of a central point from which three distinct "arms" extend, each resembling a copy of the unit interval $[0,1]$.

This simple yet elegant structure forms the foundation of the system we will be studying.

Throughout this chapter, we will take a closer look at triods, exploring their structure, properties, and significance in the broader context of our work.

Chapter five named Triod-Twist patterns, With the concept of rotation theory for triods now established, our next goal is to describe cycles that have a specific rotation number. To do this effectively, we begin by examining the most fundamental type of such a cycle: the Triod-Twist.

The Triod-Twist represents the simplest example of a cycle associated with a given rotation number, and it serves as a crucial building block for understanding more complex behavior. This pattern not only illustrates the basic principles of rotational dynamics in triods but also plays a key role in the broader study of rotation numbers. In this chapter, titled Unimodal Triod Twist Patterns, Our primary objective is to analyze the dynamics of all possible unimodal triod twist patterns associated with a specific rational rotation number.To set the stage for this discussion, we start by introducing some foundational concepts. In this chapter, titled Bimodal Triod Twist Patterns, we turn our attention to the dynamics of all possible bimodal patterns. These represent a natural extension and refinement of the unimodal case discussed earlier.

In the next chapter, we briefly summarize the conclusions of our project and outline potential directions for future research. The project explores a range of fascinating ideas and concepts in dynamical systems, presented through clear diagrams and detailed explanations. The triod is a simple yet powerful shape made up of three branches that meet at a single point. In chemistry, this structure helps us visualize how atoms connect and interact—especially in molecules where a central atom forms bonds with three surrounding atoms or groups.

A classic example is the ammonia (NH_3) molecule. In ammonia, the nitrogen atom sits at the center and forms bonds with three hydrogen atoms. If we imagine each bond as a branch, we get a perfect picture of a triod structure. Studying how this shape twists, rotates, or distorts under different conditions helps us understand the molecule's symmetry and behavior.

By thinking of the triod as something dynamic—something that can twist and change—we can explore how molecules move and evolve over time. This approach bridges the gap between static molecular shapes and the lively world of chemical reactions and transformations.

Chapter 2

Preliminaries

In this chapter we will revise some of the preliminaries from Real Analysis, Topology, Algebraic Topology and Dynamical Systems.

2.1 Real Analysis and General Topology

Definition 2.1.1. [6] (Fixed Point) A fixed point of a function g is a value x for which $g(x) = x$.

Definition 2.1.2. [4] (Topological Space) A topological space consists of a set Y together with a collection $\mathscr{T}$ of subsets of Y satisfying the following conditions:

- Both the empty set and the full set Y belong to $\mathscr{T}$; that is, $\emptyset \in \mathscr{T}$ and $Y \in \mathscr{T}$.]
- If we take an arbitrary collection of sets from $\mathscr{T}$, their union will still belong to $\mathscr{T}$.
- The intersection of a finite number of sets chosen from $\mathscr{T}$ also results in a set that lies in $\mathscr{T}$.

Definition 2.1.3. [6] (Continuous Function) A function $g : A \to B$ between two topological spaces A and B is called continuous if the preimage of every open set in B is an open set in A.

Definition 2.1.4. [6] (Lipschitz Continuous) A function f is Lipschitz continuous on a metric space (X,d) if there exists a constant L such that for all x,y in X, the following inequality holds:

$$d(f(x),f(y)) \leq L\, d(x,y)$$

Definition 2.1.5. [4] (Metric space) Suppose that Y be a non-empty set. A mapp $m : Y \times Y \to \mathbb{R}$ is called metric on Y if satisfies the following conditions:

(1) $m(p,q) \geq 0, \forall\ p,q \in Y,$

(2) $m(p,q) = 0 \Leftrightarrow p = q, \forall\ p,q \in Y,$

(3) $m(p,q) = m(q,p), \forall\ p,q \in Y,$

(4) $m(p,r) \leq m(p,q) + m(q,r), \forall\ p,q,r \in Y.$

A set Y with metric m is called metric space denoted by (Y,m).

Example 2.1.1. [6] The mapping $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ described by $m(p,q) = |p-q|, \forall\ p,q \in \mathbb{R}$ is a metric on the $\mathbb{R}$. It is also known as usual metric.

Definition 2.1.6. [4] (Homeomorphism of two topological spaces) A homeomorphism is a special type of function between two topological spaces that preserves the structure of the spaces. Specifically, a function $f : X \to Y$ between two topological spaces X and Y is called a homeomorphism if it satisfies the following conditions:

- Bijective: The function f is one-to-one (injective) and onto (surjective).
- Continuous: The function f is continuous, meaning the preimage of every open set in Y is an open set in X.
- Continuous Inverse: The inverse function $f^{-1} : Y \to X$ is also continuous.

Homeomorphic spaces are topologically equivalent, meaning they have the same topological properties even though they might look different geometrically.

Theorem 2.1.1. [6] (Intermediate Value Theorem) Let f be a continuous function on a closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$.That is, if $f(a) \leq N \leq f(b)$ or $f(b) \leq N \leq f(a)$, then there exists a number c in the interval (a,b) such that:

$$f(c) = N$$

.

Theorem 2.1.2. [6] (Mean Value Theorem) Let h be a function that satisfies the following conditions:

- h is continuous on the closed interval $[a,b]$.

- h is differentiable on the open interval (a,b).

Then there exists at least one point c in the interval (a,b) such that:

$$h'(c) = \frac{h(b) - h(a)}{b - a}$$

2.2 Dynamical System

2.2.1 Motivation

Dynamical system is the branch of mathematics which describes how a system changes with time. Dynamical systems are a branch of mathematics that help us understand how systems evolve over time. Formally, a dynamical system is built from two main components:

- A space representing all the possible states the system can take—typically a **topological space** (X, τ),
- A system's evolution over time is described by a **continuous map** $f : X \to X$, which specifies how each state transitions within the space X.

This is a rule that describes how the system evolves from one state to another over time. It's continuous, meaning that small changes in the initial state lead to small changes in the subsequent state. A dynamical system is defined as any system that changes over time according to particular rules or equations. To grasp this idea, think about a simple and enjoyable experiment with a scientific calculator. Start with any number and continuously press a function key, like the "exp" (exponential) key. This repetitive action generates a series of numbers, serving as an example of a discrete dynamical system.

Example 2.2.1. *Imagine a cup of hot coffee cooling in a room. According to Newton's Law of Cooling, the temperature of the coffee decreases over time and approaches room temperature. This system is described by the differential equation:*

$$\frac{dT}{dt} = -k(T - T_{room}) \tag{2.1}$$

where $T(t)$ is the coffee temperature at time t, and T_{room} is the ambient room temperature. This forms a continuous dynamical system with a stable equilibrium.

Example 2.2.2. 2.2.2 Predator-Prey Dynamics

In ecology, the interaction between predator and prey populations can be modeled by the Lotka–Volterra equations:

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{2.2}$$
$$\frac{dy}{dt} = \delta xy - \gamma y \tag{2.3}$$

Here, $x(t)$ is the prey population and $y(t)$ is the predator population. These equations describe how the populations influence each other over time, often leading to cyclical behavior.

Example 2.2.3. 2.2.3 Economic Supply and Demand Model

In economics, consider a model where prices adjust over time based on supply and demand. If demand exceeds supply, prices rise; if supply exceeds demand, prices fall. This can be modeled by a function $f(p)$ updating the price p at each time step:

$$p_{n+1} = f(p_n) \tag{2.4}$$

This is a discrete dynamical system. Depending on how the market reacts, the price may stabilize, oscillate, or behave unpredictably.

2.2.4 Trajectory and periodic orbit

In this study to denote the family of all real numbers we use $\mathbb{R}$ and to denote the family of all natural numbers by $\mathbb{N}$.

In the context of dynamical systems, a trajectory (or orbit) refers to the path that a point in the system follows through its state space as the system evolves over time. This concept is crucial in understanding the behavior of dynamical systems, whether they are described by differential equations, discrete maps, or other types of dynamical rules.

Definition 2.2.1. [5] (Trajectory) The series $\{u_n\}$, established iteratively as $u_{n+1} = h(u_n)$ within the function h, represents the trajectory originating from the initial point u_0 under the influence of the function h.

Definition 2.2.2. [5] (Periodic Orbit) For any x in X, if there exists a positive integer p such that $h^p(x) = h$, then x is considered a periodic point of period p. The set $\{x, h(x), h^2(x), h^3(x), ..., h^{p-1}(x)\}$ is termed a periodic orbit or cycle of period p.

Definition 2.2.3. [5] (Periodic point) A periodic point in a dynamical system is a point that returns to its original position after a fixed number of iterations of a given function. Specifically, for a function f and a point x, x is a periodic point if there exists a positive integer n such that $f^n(x) = x$, where f^n denotes the function f applied n times. The smallest such n is called the period of the point. In other words, a periodic point is a state that the system revisits exactly after n iterations.

2.2.5 Hyperbolicity

Definition 2.2.4. (Hyperbolicity) A periodic point q with a prime period m is considered hyperbolic if the condition

$$|(h^m)'(q)| \neq 1$$

is satisfied. The quantity $(h^m)'(q)$ is known as the multiplier of the periodic point.

Example 2.2.4. *Consider*

$$h(y) = \frac{1}{2}(y^3 + y)$$

The function has three fixed points at $y = 0, 1$ *and* -1.

2.2.6 Real-World Applications

Dynamical systems appear in numerous real-life contexts:

- **Biological Systems:** Neuron firings, population models, and heart rhythms.
- **Climate Modeling:** Governed by complex equations sensitive to initial conditions.
- **Engineering Systems:** Control systems in robotics or aerospace engineering often use feedback loops governed by dynamical equations to stabilize behavior or achieve goals.

Chapter 3

Sharkovskii's Theorem

3.0.1 Sharkovsky Ordering

Sharkovsky's work holds a significant place in the study of dynamical systems. To appreciate its full impact, we must first understand a unique way of ordering natural numbers—known as Sharkovsky's ordering—which forms the foundation of his groundbreaking theorem.

$$
\begin{aligned}
&3 \succeq_s 5 \succeq_s 7 \succeq_s \dots \\
2 \cdot 3 &\succeq_s 2 \cdot 5 \succeq_s 2 \cdot 7 \succeq_s 2 \cdot 9 \dots \\
2^2 \cdot 3 &\succeq_s 2^2 \cdot 5 \succeq_s 2^2 \cdot 7 \succeq_s 2^2 \cdot 9 \dots \\
&\vdots \\
2^n &\succeq_s 2^{n-1} \succeq_s 2^{n-2} \dots \succeq_s 2 \succeq_s 1
\end{aligned}
$$

If $m \succeq_s n$, then we describe m as being *sharper* than n. Let $Sh(k)$ represent the collection of natural numbers m for which the relation $k \succeq_s m$ holds, including the number k itself. Additionally, the notation $sh(2^\infty)$ is used to denote the set $\{1, 2, 4, 8, \dots\}$, which comprises all positive integer powers of 2.

3.0.2 Sharkovsky Theorem

We are now ready to explore the remarkable Sharkovsky's Theorem, a fundamental result in dynamical systems that reveals a deep connection between the existence of periodic points and the special ordering of natural numbers introduced by Sharkovsky.

Theorem 3.0.1. [2] Let $f : [0,1] \to [0,1]$ be a function that is continuous on the closed interval $[0,1]$. and $m \succ_s n$ with $m \in P(f)$. Then $n \in P(f)$; that is, if f has a periodic orbit of period m and

$m \succ_s n$, then f also has a periodic orbit of period n.

Here, $\mathrm{Sh}(k)$ denotes the set of periods ordered according to Sharkovsky's ordering.

3.0.3 Significance

The Sharkovskii theorem gives us a clear sense of how many other orbits with different periods a map will have if it has a periodic orbit with a certain period. Since it informs us of the dynamical system's complexity, this knowledge is actually quite important to us.

There is a poetic notion—one that feels almost like a whisper from Japanese philosophy—that if a butterfly takes flight in a pattern governed by rotation number 3, something unpredictable, even disastrous, may follow. While this may sound like metaphor, it echoes a profound truth in the world of dynamical systems. According to Sharkovsky's Theorem, the existence of a cycle with period 3 signals the onset of chaos; it implies the presence of cycles of every other period, unraveling the system's stability. As we venture deeper into the study of rotation numbers and triod dynamics, this idea will remain central: in the smallest cycles, great complexity may be hiding.

The number 3 holds a special place in Sharkovsky's ordering—it appears near the very beginning, signifying its powerful role in the behavior of dynamical systems. If a system, described by a function f, contains a periodic orbit of period 3, then Sharkovsky's Theorem tells us something profound: the function f must also have periodic orbits of every other natural number. In other words, the presence of a period-3 cycle unlocks a cascade of complexity within the system. When this happens, we say the system is *chaotic*, as it can exhibit highly intricate and unpredictable behavior.

Chapter 4

Rotation Theory

4.1 Motivation and need

If a continuous interval map has a periodic orbit of period 9, it will also have a periodic orbit of $6, 12, 20, 40, \ldots$. In particular, if $f : [0,1] \to [0,1]$ has a periodic orbit of period 3, it has periodic orbits of all periods. Sharkovsky's theorem categorizes periodic orbits based on their period, but this isn't a very precise method because many cycles can has in common period. A more detailed approach involves examining the "cycle permutation," which captures the way a map rearranges the points of a cycle when they are listed in left-to-right order. However, this method proved to be overly intricate and didn't provide clear insights. Consequently, "Rotation Theory" was developed as an intermediate solution for classifying cycles, offering a balance between simplicity and detail.

In the next chapter, we will explore rotation theory in detail, uncovering its key ideas, principles, and applications.

4.2 Rotation Theory in a general topological space

Definition 4.2.1. Let X be an arbitrary compact topological space. Let $\phi : X \to \mathbb{R}$ be a continuous function(observable). Let $f : X \to X$ be a continuous map.

Definition 4.2.2. Let $x \in X$. The set of all subsequential limits of the sequence

$$\left(\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) \right)$$

is called the ϕ-rotation set of x, denoted by $\rho_\phi(x)$.

If $\rho_\phi(x) = \{\rho\}$ is a singleton, then ρ is called the *rotation number* of x.

The ϕ-rotation set of the function f, denoted by $\rho(f)$, is defined as the union of the ϕ-rotation sets corresponding to each point in X.

Definition 4.2.3. If x is a periodic point of f with period n, then its rotation number is well-defined and given by

$$\rho(x) = \frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i(x)).$$

Since every point in the orbit of x has the same rotation number, this value is called the *rotation number of the cycle P* and denoted by $\rho(P)$.

4.2.1 Rotation Pair

We now introduce a new concept called the **rotation pair**. For a periodic point x of period n, the pair

$$\left(\sum_{i=0}^{n-1}\phi(f^i(x)),\ n\right)$$

is referred to as the rotation pair of P. It's going to be a big deal for us later when we dive into how things works on rotation theory.

4.2.2 lifting

A circle map is defined as a continuous function from the unit circle to itself. Let $f : S^1 \to S^1$ denote such a map. Consider the projection map

$$\pi : \mathbb{R} \to S^1$$

given by

$$\pi(t) = e^{2\pi it}.$$

Then, there exists a function $F : \mathbb{R} \to \mathbb{R}$ such that

$$f \circ \pi = \pi \circ F.$$

This function F is called a *lifting* of the circle map f.

4.2.3 Property of Lifting

- If $f : S^1 \to S^1$ is a circle map and $F : \mathbb{R} \to \mathbb{R}$ is its lifting, then there exists an integer d such that $F(x+1) = F(x) + d \ \forall x \in \mathbb{R}$. This number d is called the degree of the lifting map F.
- Let $\mathscr{L}_1$ be the collection of all degree one lifting of circle maps $F(x+k) = F(x) + k * 1$ $\forall k \in \mathbb{Z}$

4.3 Rotation number of circle maps

Let $\mathscr{L}_1$ be the collection of all degree one lifting maps. Let $F \in \mathscr{L}_1$ and x be a real number.

Define an upper-rotation number of $F \in \mathscr{L}_1$ at $x \in \mathbb{R}$

$$\bar{\rho}_F(x) = \limsup_{n\to\infty}\left(\frac{F^n(x) - x}{n}\right)$$

And lower-rotation number of $F \in \mathscr{L}_1$ at $x \in \mathbb{R}$

$$\underline{\rho}_F(x) = \liminf_{n\to\infty}\left(\frac{F^n(x) - x}{n}\right)$$

If $\bar{\rho}_F(x)$=$\underline{\rho}_F(x)$=$\rho_F(x)$
then we say that $\rho_F(x)$ is the rotation number of F at x.

4.3.1 Properties of Rotation Numbers for Circle Maps

Theorem 4.3.1. *Consider a map $f : S^1 \to S^1$ defined on the circle. Let $F : \mathbb{R} \to \mathbb{R}$ be a function that serves as a lift of f to the real line. Suppose there exist $x \in \mathbb{R}$ and $n \in \mathbb{N}$ such that*

$$f^n(\pi(x)) = \pi(x),$$

where $\pi : \mathbb{R} \to S^1$ is the natural projection. Then,

$$F^n(x) = x + k$$

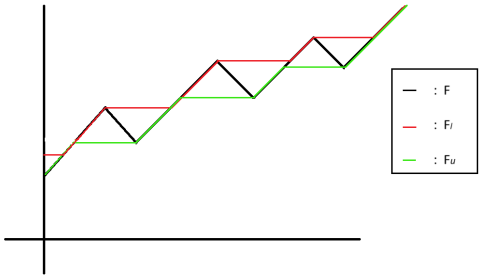


Figure 4.1: F_l and F_u

for some integer $k \in \mathbb{Z}$, and the rotation number of x under F is

$$\rho_F(x) = \frac{k}{n}.$$

Theorem 4.3.2. *Let $\mathscr{L}_1'$ represent the set of all non-decreasing functions contained in L_1. Suppose $F \in \mathscr{L}_1'$ is a lift of a circle map f. Then, for each $x \in \mathbb{R}$, the rotation number $\rho_F(x)$ is well-defined and remains constant regardless of the choice of x. Furthermore, $\rho_F(x)$ is a rational number precisely when the map f possesses a periodic point. This constant value is referred to as the rotation number of F, denoted $\rho(F)$.*

Definition 4.3.1. (rotation interval) The rotation interval of a degree one circle map F is the closure of the collection of all rotation numbers of F.

4.3.2 F_l and F_u map

Let $F : \mathbb{R} \to \mathbb{R}$. Construct the following sets-

$$F_\ell(x) = \inf\{F(y) | y \geq x\}$$

$$F_u(x) = \sup\{F(y) | y \leq x\}$$

To determine the rotation interval, we make use of two important functions: the lower map F_l and the upper map F_u. These maps help us capture the range of rotation numbers that can arise in the system.

Theorem 4.3.3. *Let $F \in \mathscr{L}_1$ is a lifting of a circle map f .Then the set of all rotation numbers of points is equal to -*

$$[\rho(F_\ell), \rho(F_u)]$$

This interval $[\rho(F_\ell), \rho(F_u)]$ is called the rotation interval of the map and is denoted by $Rot(F)$.

Applications of Rotation Theory in Circle Maps

- **Understanding Long-Term Behavior:** Rotation theory provides a framework for determining whether a dynamical system exhibits periodic behavior, quasi-periodicity, or chaotic dynamics over time.
- **Explaining Real-World Systems:** It is commonly applied to the analysis of systems exhibiting cyclical behavior, such as pendulums, electrical circuits, and biological rhythms—essentially, any system characterized by rotational or repetitive dynamics over time.
- **Predicting Sudden Changes:** It provides a framework for identifying transitions in dynamical behavior, such as when a system locks into a periodic rhythm or shifts toward chaotic dynamics.
- **Simplifying Complex Patterns:** Rotation theory simplifies complex dynamical motion by translating it into more tractable patterns, thereby facilitating analysis and deeper understanding.

4.4 Over-Rotation theory for Interval Maps

4.4.1 Introduction

In the last chapter we understand the rotation theory in general. We also study rotation theory of degree one maps of cycle.In this current chapter we will extend rotation theory to the maps on the interval.

Definition 4.4.1. Let $g : [0,1] \to [0,1]$ be a continuous function. Consider a periodic orbit P of period n. Define a function $\theta : [0,1] \to \mathbb{R}$ by

$$\theta(x) = \begin{cases} \frac{1}{2}, & \text{if } (g(x)-x)(g^2(x)-g(x)) \leq 0, \\ 0, & \text{if } (g(x)-x)(g^2(x)-g(x)) > 0. \end{cases}$$

Then, the set of all limit points of the sequence

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \theta(g^i(x)) \right\}$$

is called the θ-rotation set of the point x, denoted by $\rho_\theta(x)$. If this set is a singleton $\{\rho\}$, then ρ is called the *Over-Rotation Number* of x.

4.4.2 Over-rotation number for a periodic point

Definition 4.4.2. Let P be a periodic orbit of period n. Let m be the number of points $x \in P$ such that

$$(f(x)-x)(f^2(x)-f(x)) \leq 0.$$

Then, the number $\frac{m}{2}$ is called the *over-rotation number* of P, and the pair $\left(\frac{m}{2}, n\right)$ is called the *over-rotation pair* of P.

Theorem 4.4.1. *[2] If $f : [0,1] \to [0,1]$ has a unique fixed point then the over-rotation number simplifies to $\frac{p}{q}$ Here, p denotes the number of elements located to the right of the fixed point that are sent to the left side under the mapping, while q represents the period associated with the periodic orbit.*

4.4.3 Over-rotation Interval

Theorem 4.4.2. [2] If $f : [0,1] \to [0,1]$ is a continuous map and f has a periodic orbit of over-rotation number $\frac{p}{q}$, and if $\frac{r}{s} < \frac{p}{q}$, then f must have a periodic orbit with over-rotation number $\frac{r}{s}$.

Definition 4.4.3. [2] (Over-rotation Interval) The set of over-rotation numbers associated with the cycles of the map f, upon taking its closure, yields the interval $I_f = [r_f, \frac{1}{2}]$. This interval is termed the *over-rotation interval* of f.

This result demonstrates a density property for over-rotation numbers: whenever there is a periodic orbit with a certain over-rotation number, periodic orbits exist for every smaller rational over-rotation number as well.

4.4.4 Over-twist patterns

Once we understand the idea behind over-rotation theory, a natural next question arises: Given a pattern π, can we determine all the patterns it forces? To explore this, we need to delve into an important concept known as the over-twist pattern.

Definition 4.4.4. Let us consider a pattern π. Let I_π be te forced over-rotation interval of π. Let I_π=$[r_\pi, \frac{1}{2}]$.
Then,

$$r_\pi \leq \rho(\pi)$$

A pattern π is said to be an *over-twist pattern* if it satisfies $r_\pi = \rho(\pi)$ and has a co-prime over-rotation pair.

4.4.5 Finding the Over-Rotation Interval

Theorem 4.4.3. *Let $f : [0,1] \to [0,1]$ be a continuous map with one or two critical points. Then, there exists a monotonically increasing, degree-one map G_f such that the rotation number of G_f determines the left endpoint r_f of the over-rotation interval:*

$$r_f = \rho(G_f).$$

$$I_f = [r_f, \tfrac{1}{2}].$$

Proof. We can assume, without loss of generality, that f has a unique fixed point a. This is because Blokh and Misiurewicz (1998) proved that if a function f has more than one fixed point, then $I_f = [0, \frac{1}{2}]$, i.e., it has periodic orbits of all possible over-rotation numbers and periods—indicating that it is a chaotic map.

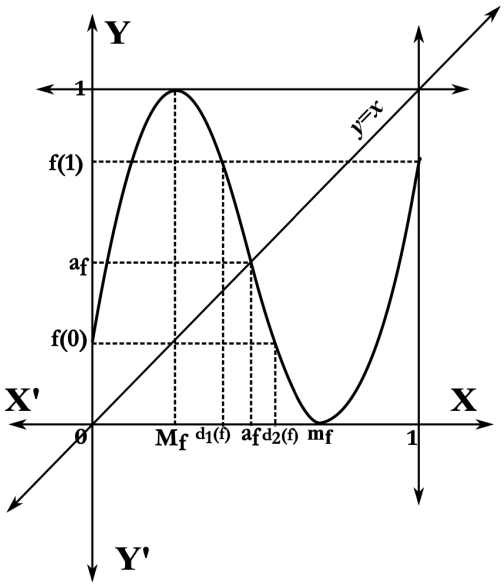


Figure 4.2: Bimodal map

We define the map

$$\sigma_f(x) = \begin{cases} x, & \text{if } 0 \leq x < a_f, \\ a_f + 1 - x, & \text{if } a_f \leq x \leq 1. \end{cases}$$

It is easy to see that $\sigma_f^{-1}(x) = \sigma_f(x)$.

Now consider the map $g_f = \sigma_f^{-1} \circ f \circ \sigma_f$, i.e., g_f is conjugate to the map f. Since σ_f is an involution, we can also write:

$$g_f = \sigma_f \circ f \circ \sigma_f.$$

- **Case 1:** If $x \in [0, a'_f]$, $\sigma_f(x) = x$,
  $f(\sigma_f(x)) = f(x)$,
  $\sigma_f(f(\sigma_f(x))) = f(x)$
  $\implies g(x) = f(x)$

- **Case 2:** If $x \in [a'_f, a_f]$, $\sigma_f(x) = x$,
  $f(\sigma_f(x)) = f(x) \in [a_f, 1]$,
  $\sigma_f(f(\sigma_f(x))) = a_f + 1 - f(x)$

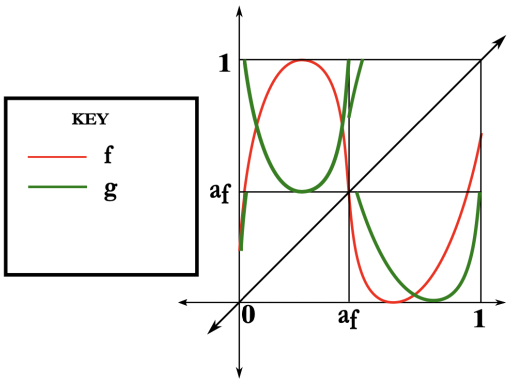


Figure 4.3: Interpretation of f and g_f

$\implies g(x) = a_f + 1 - f(x)$ i.e.,graph of g is obtined from graph of f by reflecting vertically across $y = \frac{1+a_f}{2}$.

- **Case 3:** If $x \in [a_f, a_f'']$, $\sigma_f(x) = a_f + 1 - x$,
  $f(\sigma_f(x)) = f(a_f + 1 - x)$,
  $\sigma_f(f(\sigma_f(x))) = f(a_f + 1 - x)$

  $\implies g(x) = f(a_f + 1 - x)$ i.e.,graph of g is obtined from graph of f by reflecting across the line $x = \frac{1+a_f}{2}$.

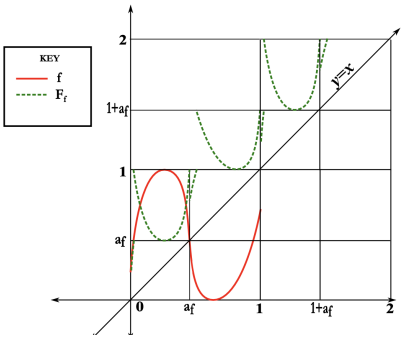
- **Case 4:** If $x \in [a_f'', 1]$, $\sigma_f(x) = a_f + 1 - x$,
  $f(\sigma_f(x)) = f(a_f + 1 - x)$,
  $\sigma_f(f(\sigma_f(x))) = a_f + 1 - f(a_f + 1 - x)$

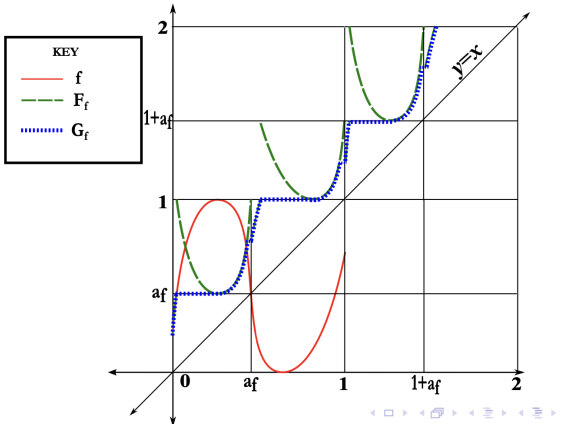
  i.e.,graph of g in $[a_f'', 1]$ is obtined first by reflecting horizontally the graph of f across from graph of f by reflecting across $x = \frac{1+a_f}{2}$,then across reflecting verticall the graph of f across $y = \frac{1+a_f}{2}$.

We now define the map

$$F_f(x) = \begin{cases} g(x) + 1, & \text{if } x > a_f \text{ and } g(x) < a_f, \\ g(x), & \text{otherwise.} \end{cases}$$

We now extend $F_f(x)$ from $[0,1]$ to $\mathbb{R}$ using $F_f(x+1) = F_f(x) + 1$ now, by construction the over-rotation numbers of points under$f$ gives us the classical rotation number of the degree one map F_f now, we define the map $G = inf F_f | y \geq x$ G is monotonically non-decreasing. so, it's rotation number at each point is same. Let it be ρ Then, ρ by construction is less then rotation number of





each point under F_f. Combining, we get $\rho = r_f$ So,the over rotation number of G is the left end point r_f. Over-rotation interval of $f = [r_f, \frac{1}{2}]$

=Collection of all over-rotation number of all points under f

□

4.5 Concluding remarks

A substantial part of this chapter is dedicated to outlining an algorithm for determining the left endpoint r_f of the over-rotation interval. This method employs a conjugacy of the original map to construct a monotone degree-one lift G_f, whose classical rotation number coincides with r_f.

The transformation sequence—from the original map f to the symbolic map g_f, then to the lifted map F_f, and ultimately to G—establishes a bridge between symbolic over-rotation theory and classical circle rotation theory.

This framework offers not only a practical computational tool but also a conceptual unification of symbolic and geometric perspectives in dynamical systems. The accompanying graphs throughout the section visualize each transformation step and clarify the internal structure of the maps involved.

Overall, this approach deepens our understanding of interval dynamics by showing how

local turning behavior aggregates into global rotational motion, providing a powerful lens through which to analyze the complexity of continuous interval maps.

Chapter 5

Triods

5.1 Introduction

An n-od can be described as the collection of complex numbers $z \in \mathbb{C}$ such that z^n lies within the interval $[0,1]$. Geometrically, this structure features a central node with n branches radiating outward, where each branch is similar to a unit segment extending from the center. The special case $n = 3$ corresponds to a triod, characterized by exactly three branches radiating from the central point.

In [3], it was noted that the collection of periods of periodic orbits for such maps can be described as unions of initial segments of a linear ordering on the rational numbers in the interval $(0,1)$, with denominators bounded by n. However, these observations were mostly empirical and lacked a rigorous theoretical framework.

Building on this foundation, our current goal is to establish a formal approach that yields analogous results, concentrating on maps whose dynamics correspond to these initial behaviors.

Let us represent the triod by T, with its central junction point labeled as a. Each connected component that arises from removing the point a from T, denoted $T \setminus \{a\}$, is called a branch of T. The point a is known as the branching point.

A map $P : T \to T$ is said to be P-linear with respect to a cycle P if it fixes the point a and is affine on each connected component of the convex hull $[P]$, excluding the points in $P \setminus \{a\}$.

We define $\mathscr{U}$ as the family of continuous maps from T into itself for which the central point a is the unique fixed point.

5.2 Ordering of points

We write $x > y$ if x and y lie on the same branch of the triod T, and x is farther from the central point a than y is.

5.3 Primitive patterns

A cycle and its associated pattern are termed *primitive* if every point in the cycle lies on a different branch of the triod T.

Theorem 5.3.1. *Let f be a P-linear map such that the cycle P displays the pattern $\mathscr{A}$. Then, the pattern $\mathscr{B}$ is said to be forced by $\mathscr{A}$ precisely when f admits a cycle Q whose pattern coincides with $\mathscr{B}$.*

5.4 Definitions

Definition 5.4.1 (Pattern)**.** The *pattern* of a cycle P is defined as the equivalence class to which P belongs.

Definition 5.4.2. Let $f \in \mathscr{U}$ be a map. A cycle P is said to exhibit the pattern $\mathscr{A}$ if P belongs to the equivalence class represented by $\mathscr{A}$.

Definition 5.4.3. We say that a pattern $\mathscr{A}$ *forces* another pattern $\mathscr{B}$ if, for every map $f \in \mathscr{U}$ that contains a cycle exhibiting $\mathscr{A}$, there also exists a cycle whose pattern is $\mathscr{B}$.

5.5 Oriented Graph

Let $f \in \mathscr{U}$ and $P \subset T \setminus \{a\}$ be a finite set. We define the *oriented graph* G_P associated with P as the directed graph whose vertices correspond to the points in P. For any two points $x, y \in P$, there is an arrow $x \to y$ if and only if there exists a point $z \in T$ satisfying

$$x > z \quad \text{and} \quad f(z) > y.$$

Theorem 5.5.1. *The following hold:*

(1) Suppose there is a loop in the graph G_P of the form

$$x_0 \to x_1 \to \cdots \to x_{m-1} \to x_0.$$

Then there exists a point $y \in X \setminus \{a\}$ such that $f^m(y) = y$, and for each $k = 0, 1, \ldots, m-1$, the points x_k and $f^k(y)$ lie on the same branch of X.

(2) Let f be a P-linear map for some cycle $P \neq \{a\}$. If $y \neq a$ is a periodic point of f with period q, then there exists a loop

$$x_0 \to x_1 \to \cdots \to x_{q-1} \to x_0$$

in G_P such that $x_i > f^i(y)$ for all i.

5.6 Rotation Number for Triods

Next, we focus on the notion of the *rotation number* for triods, extending the ideas previously developed for interval maps.

Denote by $\mathscr{A}$ the collection of all directed edges (arrows) in the oriented graph G_P.

We label the branches of the triod T in an anticlockwise manner as follows:

$$\mathscr{B} := \{b_i \mid i \in \{0,1,2\}\},$$

where the addition in the subscript of b is taken modulo 3, so that each b_i corresponds to a branch of T. We define a function referred to as the *displacement function*.

$$\psi : \mathscr{B} \times \mathscr{B} \to \mathbb{R}$$

by

$$\psi(b_i, b_j) = \frac{k}{3},$$

where $j \equiv i + k \pmod 3$.

This function Ψ naturally induces a *displacement function*

$$\widehat{d} : \mathscr{A} \to \mathbb{R}$$

defined on the full collection of arrows $\mathscr{A}$ in the oriented graph G_P. For $u, v \in T$, the displacement is defined as

$$\widehat{d}(u \to v) = \Psi(b_i, b_j),$$

where $u \in b_i$ and $v \in b_j$.

For a point loop Γ' in G_P, the total displacement along the loop is denoted by

$$\widehat{d}(\Gamma') = \sum_{(u \to v) \in \Gamma'} \widehat{d}(u \to v).$$

In our model of T, this value represents the total number of times the loop winds around the origin in the counterclockwise direction, and hence $\widehat{d}(\Gamma') \in \mathbb{Z}$.

The *rotation pair* and the corresponding *rotation number* associated with the loop Γ' are given by

$$\mathrm{rp}(\Gamma') = \big(\widehat{d}(\Gamma'), |\Gamma'|\big), \quad \rho(\Gamma') = \frac{\widehat{d}(\Gamma')}{|\Gamma'|},$$

where $|\Gamma'|$ denotes the total count of arrows within the loop.

The *rotation set* of the graph G_P, denoted by $\mathscr{L}(G_P)$, is defined as the closure of all rotation numbers corresponding to loops in G_P. As established in [**?**], this set is exactly the smallest interval that includes the rotation numbers of all elementary loops within the graph.

5.7 Modified Rotation Number

We define the notion of a *modified rotation pair* associated with a cycle P, constructed in analogy with the traditional rotation pairs used in the context of interval and circle mappings. For a given pattern A, let $\mathrm{mrp}(A)$ represent the collection of all such modified rotation pairs that are either implied or determined by this pattern.

To illustrate, suppose a point x has a rotation pair $\mathrm{rp}(x) = (mp, mq)$, where p and q are coprime integers and $m \in \mathbb{N}$. In this case, we reinterpret the pair as (t, m), where

$$t = \frac{p}{q}$$

is referred to as the *modified rotation number*.

Theorem 5.7.1. *If P and Q are two periodic orbit with mrp $(\frac{p}{q}, m)$ and $(\frac{r}{s}, n)$ respectively, then we have two cases*

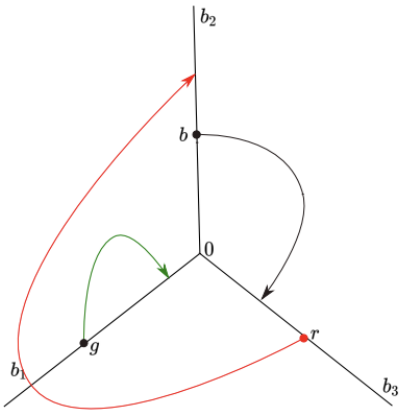


Figure 5.1: Color of Points

case1 *If* $\frac{p}{q} < \frac{r}{s} < \frac{1}{2}$ *then P forces Q.*

Case2 $\frac{p}{q} = \frac{r}{s}$ *then then P forces Q if* $m \succeq_s n$ *by Sharkovsky's ordering.*

5.8 Regular pattern

In our analysis, the idea of a *regular pattern* is pivotal. A pattern $\mathscr{A}$ associated with a map $f \in \mathscr{U}$ is called *regular* if it does not enforce the presence of any primitive pattern of period two. Correspondingly, a cycle P is said to be *regular* if it realizes a regular pattern.

Theorem 5.8.1. *[3] Let* $\mathscr{A}$ *be a regular pattern associated with a map* $f \in \mathscr{U}$*. Then there exist patterns* $\mathscr{B}$ *and* $\mathscr{C}$ *with modified rotation pairs* (t_1, m_1) *and* (t_2, m_2)*, respectively, such that*

$$\mathrm{mrp}(\mathscr{A}) = [(t_1, m_1), (t_2, m_2)].$$

5.9 Color of Points

When examining the directed paths in the oriented graph G_P, where $P \subset T \setminus \{a\}$ is finite, we apply the color coding introduced in [**?**]. Each arrow $u \to v$ in G_P, with $u, v \in P$, is assigned a color — *green*, *black*, or *red* — based on its displacement value $d(u \to v)$, which takes values 0, $\frac{1}{3}$, or $\frac{2}{3}$, respectively.

If P be a cycle of a map $f \in \mathscr{U}$, then each point $x \in P$ is assigned a color corresponding to the color of the directed edge $x \to f(x)$ in its associated with the fundamental loop.

Theorem 5.9.1. *[3]*

Theorem 5.9.2. *Let P be a regular cycle. Then, for every point $x \in P$, there exists a black loop of length 3 that contains x.*

Chapter 6

Triod-Twist Patterns

Definition 6.0.1. A triod twist is a specific kind of pattern. What makes it special is that it doesn't force any other different patterns to exist if those patterns have the same "rotation number" – which is a way of describing their overall rotational movement. Essentially, a triod twist is a unique and fundamental pattern within its rotational group.

Our goal is to figure out exactly what makes a regular pattern count as a triod twist. To get there, we look closely at the features of these patterns and notice that something changes in their behavior—something that depends on the rotation number. This change, or bifurcation, marks a shift in how the patterns behave. But before we get into that, we need to understand the next important lemma.

6.1 Properties of regular cycles

Before we can really understand the idea of a triod-twist, we first need to go through a few important lemmas.

Lemma 6.1.1. *[3] In a regular cycle P,every green point is directed or transformed inward.*

Let f be a P-linear map. Assume that there exists a green point $x \in P$ such that it is mapped outward, meaning $f(x) > x$. Since the set of points $\{z \in P : z > x\}$ is not invariant under f, there exists a point $y \in P$, $y > x$, such that either

(i) $y > f(y)$, or

(ii) $f(y)$ lies in a branch different from that of y.

In both scenarios, the map f must have a fixed point lying inbetween x and y, which leads to a contradiction.

Lemma 6.1.2. *[3]A regular cycle P, every branch of the structure has at least one point.*

Proof. Assume that the cycle P is fully contained within a single branch b of the tree T. Let x denote the point in P closest to the branching point a, and let y be the point farthest from a along the same branch. Since $f(x) > x$ and $f(y) \leq y$, the continuity of f combined with the Intermediate Value Theorem ensures the existence of a fixed point $a' \neq a$ located between x and y. This outcome contradicts the premise that P is a regular cycle.

Next, suppose P lies on exactly two branches of T. Let u and v be the points in P closest to a on these branches, respectively. According to Lemma 7.1.1, it holds that $f(u) > v$ and $f(v) > u$, which yields a 2-cycle $u \to v \to u$ in the directed graph G_P. Consequently, P must force a primitive cycle of period two, contradicting the assumption that P is regular.

□

Lemma 6.1.3. *[3] If a cycle P is regular, then it necessarily implies the forcing of a primitive cycle of period 3.*

Proof. Every loop in the graph G_P must visit each branch at least once. Consider the shortest loop γ in G_P. Suppose γ contains two distinct arrows, A_1 and A_2, both terminating in the same branch, at points x_1 and x_2, respectively, with $x_1 > x_2$.

We can replace the arrow A_1 by a new arrow A_3 that starts at the same vertex as A_1 but ends at x_2. This substitution yields a loop shorter than γ. By repeating this reduction process a finite number of times, we obtain a loop γ' where no two arrows terminate in the same branch.

According to Proposition 4.0.2, the resulting loop γ' forces period 3 primitive cycle.

□

Lemma 6.1.4. *[3] Whenever a triod twist pattern π is associated with a rotation pair (p,q), it turns out that p and q can't have any common divisors except 1*

Proof. Let P be a cycle that exhibits the pattern π, and suppose that the integers p and q are not coprime; that is, they have a common divisor greater than one. Then there exist coprime integers s and t such that

$$\frac{p}{q} = \frac{s}{t}, \quad \text{and let} \quad u = \frac{p}{s} = \frac{q}{t} > 1.$$

Thus, the modified rotation pair of P is given by $\left(\frac{s}{t}, u\right)$. According to Theorem 1.9, this implies that P must force a cycle with modified rotation pair $\left(\frac{s}{t}, 1\right)$, corresponding to a cycle with rotation pair (s,t).

Consequently, P forces another cycle $Q \neq P$ sharing the same rotation number, contradicting the regularity of P.

□

Lemma 6.1.5. *[3] If P is a regular cycle, then the points p_0, p_1, and p_2 are consistently colored; they are either all black or all red.*

Proof. Suppose p_0 is black. Then $f(p_0) > p_1$. If p_1 is red, then $f(p_1) \in b_0$, and hence $f(p_1) > p_0$. This creates a loop $p_0 \to p_1 \to p_0$ in the graph G_P. By Theorem 4.0.2, this loop forces a primitive cycle of period 2, contradicting the regularity of P. Therefore, p_1 must be black.

By similar reasoning, if p_1 is black, then p_2 must also be black. Thus, if p_0 is black, all three points p_0, p_1, p_2 are black.

Conversely, if p_0 is red, a symmetric argument shows that p_1 and p_2 must also be red. □

Lemma 6.1.6. *[3] Among triod twist patterns, π_3 uniquely has a rotation number of $\frac{1}{3}$.*

Proof. Consider a cycle P that follows the pattern π, and let f be a map adjusted to P. Assume, for contradiction, that P has a red point x.

By Lemma 3.4, the period of P must be 3. Therefore, $m_1 < 2$, since there cannot be more than one green (and hence red) point in a cycle of period 3 with $m_1 = m_3$. From Lemma 3.5, it then follows that $m_1 = m_3 = 0$ and $m_2 = 3$.

Thus, P consists entirely of black points and has period 3, implying it exhibits the pattern π_3. Hence, π_3 is the unique triod twist pattern with rotation number $\frac{1}{3}$. □

Lemma 6.1.7. *[3] If a triod twist pattern π has a rotation number less than $\frac{1}{3}$, then it does not have any red points.*

Proof. Let P be a cycle exhibiting the pattern π, and let f be a P-adjusted map. Suppose, for contradiction, that the cycle P contains a red point x.

By Theorem 4.0.5, there exists a black f-point loop $x \to x_1 \to x_2 \to x$ of length 3 passing through x. This implies that $y = f(x)$ and x_2 lie on the same branch of the tree T. Thus, either $y > x_2$ or $x_2 > y$.

If $y > x_2$, then there exists a shorter f-point loop $x \to x_2 \to x$. By Theorem 1.6, this implies that P forces a primitive cycle of period 2, contradicting the regularity of P.

Suppose that $x_2 > y$, and let γ be the fundamental loop associated with the periodic orbit P. Given this inequality, the segment $x \to y$ in γ can be replaced by the path $x \to x_1 \to y$, yielding a new loop γ'.

The loop γ' is strictly longer than γ, which implies that its rotation number ρ' satisfies $\rho' < \rho$. By Theorem 4.0.2, the pattern P forces the existence of a cycle R with rotation number fulfilling

$$\rho' < \rho < \frac{1}{3}.$$

Moreover, Theorem 4.0.5 and Lemma 3 guarantee the existence of a cycle $Q \neq P$ of the map f with the identical rotation number ρ, contradicting the assumption that P is a triod-twist cycle.

Therefore, P cannot contain any red points. . □

Lemma 6.1.8. *[3] A triod twist pattern π whose rotation number ρ satisfies*

$$\rho > \frac{1}{3}$$

does not contain any green points.

Proof. Let P be a cycle realizing the pattern π, and let f be a map adapted to P. Suppose, for the sake of contradiction, that P contains a green point x.

By Lemma 7.1.5, it holds that $x > f(x)$. Consider the fundamental loop δ associated with P. Removing the arrow $x \to f(x)$ from δ yields a new loop δ' whose rotation number is strictly greater than ρ.

According to Theorem 4.0.2, the map f admits a cycle U with rotation number larger than ρ. Furthermore, Theorem 4.0.5 together with Lemma 3 guarantees the existence of another cycle $S \neq P$ sharing the rotation number ρ, contradicting the uniqueness of P.

Therefore, P cannot contain any green points.

□

From last three lemma, we can clearly see that the rotation number $\rho = \frac{1}{3}$ plays a critical role in determining the behavior of triod-twist patterns—it marks a bifurcation point.

We have now obtained a full characterization of triod twist patterns, which we will shortly modify slightly to apply them within the framework of triod maps.

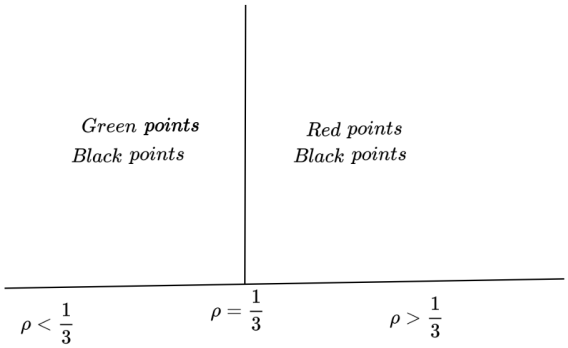


Figure 6.1: Bifurcation diagram illustrating how point colors change in response to variations in the rotation number ρ.

6.2 Green Pattern

We say that a regular cycle P is *green* if for any two points $x, y \in P$ with $x > y$, whenever $\{(x)$ and $\{(y)$ lie on the same branch of the tree T, it follows that

$$\{(x) > \{(y).$$

A pattern π is said to be *green* if every cycle exhibiting π satisfies the aforementioned condition.

6.3 Code Function

Definition 6.3.1. [3] Let P be a cycle of a map $f \in \mathscr{R}$ with rotation number $\rho \neq \frac{1}{3}$. We define the *code function* $\Psi : P \to \mathbb{R}$ as follows:

1. Choose a point $x_0 \in P$ and set $\Psi(x_0) = 0$.

2. For each integer k, define

$$\Psi(f^k(x_0)) = \Psi(x_0) + k\rho - \lfloor t_k \rfloor,$$

where

$$t_k = \sum_{j=0}^{k-1} d\big(f^j(x_0), f^{j+1}(x_0)\big).$$

Here, $\lfloor x \rfloor$ denotes the floor function, i.e., the greatest integer less than or equal to x. The value $\Psi(x)$ is called the *code* of the point $x \in P$.

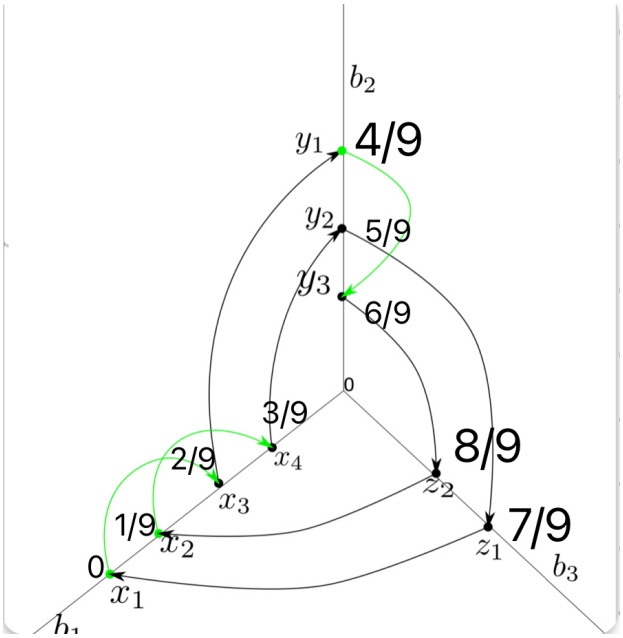


Figure 6.2: Finding the codes of the points

6.4 Non-decreasing Phase Function

Definition 6.4.1. Let P be a cycle having rotation number $\rho \neq \frac{1}{3}$, and let Ψ represent the corresponding code function.

We say that Ψ is **non-decreasing** if it satisfies the following conditions:

1. If the rotation number ρ is less than $\frac{1}{3}$, then for any points x, y with $x > y$, the inequality $\Psi(x) \leq \Psi(y)$ holds.

2. If the rotation number ρ is greater than $\frac{1}{3}$, then for any points x, y with $x > y$, we have $\Psi(x) > \Psi(y)$.

Phase function Ψ of a cycle P is termed **strictly increasing** if it is monotonically non-decreasing and any two adjacent points of P within the same branch of the tree have distinct phase values.

Example 6.4.1. *By using the the definition of code and ket the code for $x_1 = 0$ then finds the other codes iteratively*

Theorem 6.4.1. [2] A regular pattern π qualifies as a *triod twist* precisely when it is **green** and its code function Ψ increases strictly along the cycle and vice-versa.

6.5 Unimodal Triod-Twist Patterns

We now focus on describing the dynamics of each *unimodal triod twist pattern* corresponding to a fixed rational rotation number. To lay the groundwork, we start by introducing several key definitions.

A group of consecutive points sharing the same color and located on the same branch is called a *block* of that color. Accordingly, such blocks may be classified as blocks that are green, blocks that are black, or blocks that are red.

Consider two subsets A and B of the triod T. We say $A > B$ if every element in A is strictly greater than each element in B. For a unimodal cycle P, it is possible for there to be at most one block C, potentially of a distinct color, such that C lies entirely above some block B_j, where $j \in \{1,2,3\}$. Any other blocks that are not strictly greater than B_j must lie within the remaining two branches, b_{j+1} and b_{j+2}, with indices taken modulo 3.

Assume the distinct block C is green. Let $b = |B_j|$ denote the number of points in the block B_j, and let $g = |C|$ represent the number of points in C. Since the mapping carries B_j onto B_{j+1} in an order-preserving manner, it follows that $|B_{j+1}| = b$. By the same argument, we also have $|B_{j+2}| = b$.

Case of Rotation Number $< \frac{1}{3}$

[3]A straightforward calculation reveals that $b = r$ and $g = s - 3r$ since P's rotation number is $\frac{r}{s}$. As a result, $s-2r$, r, and r points are contained in the branches b_j, b_{j+1}, and b_{j+2} (with addition in the subscript taken modulo 3).

We label these points as follows:

- $x_1, x_2, \ldots, x_{s-2r}$
- $y_1, y_2, \ldots, y_r$
- $z_1, z_2, \ldots, z_r$

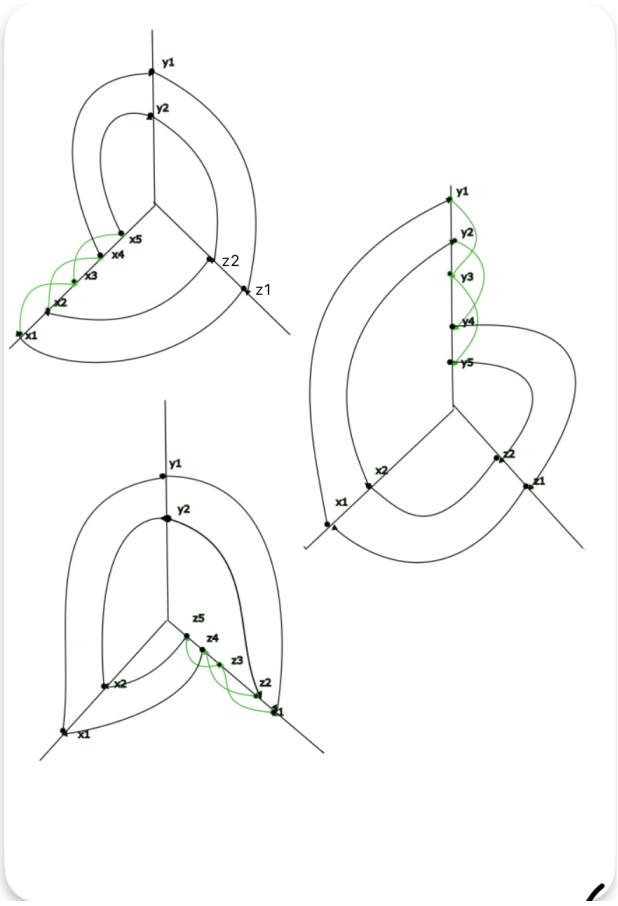


Figure 6.3: $\Gamma^i_{2/9}$ for $i = 1,2,3$ corresponding to $\rho = \frac{2}{9} < \frac{1}{3}$.

(1) The r points closest to a along the branch b_j are mapped in an order-preserving manner to r points on branch b_{j+1}. These points on b_{j+1} are then similarly mapped in order to r points on branch b_{j+2}.

Simply put, for each $i = 1,2,\ldots,r$, we have:

$$f(x_i) = y_i \quad \text{and} \quad f(y_i) = z_i.$$

(2) The final point x_{s-2r} on the branch b_j, Counted outward from the branching point a, these points are the images of specific elements from another branch. b_{j+2} (with indices considered modulo 3). It follows that:

$$f(z_r) = x_{s-2r}.$$

Since the pattern P is unimodal, the images of points on the branch b_{j+2} do not spread out when mapped to branch b_j. Consequently, the r black points z_i on b_{j+2} are mapped onto

the last r points of b_j in an order-preserving way.

(3) Therefore, the remaining points $x_1, x_2, \ldots, x_{s-3r}$ on the branch b_j must be the images of the green points $x_r, x_{r+1}, \ldots, x_{s-2r}$, which also lie on the same branch.

Case of Rotation Number $> \frac{1}{3}$

[3]We now examine the situation where the rotation number $\frac{r}{s}$ is greater than $\frac{1}{3}$. In this situation, the block C is of red color.

Let $|C| = r_1$ and $|B_j| = b$. Consequently, the calculation yields:

$$b = s - 2r \quad \text{and} \quad r_1 = 3r - s.$$

As in the previous setup, we assign labels to the points in the set P located along branches b_j, b_{j+1}, and b_{j+2}, ordering them outward starting from the central point a.

- $x_1, x_2, \ldots, x_r$ in branch b_j,
- $y_1, y_2, \ldots, y_{s-2r}$ in branch b_{j+1},
- $z_1, z_2, \ldots, z_r$ in branch b_{j+2}.

(1) The initial $s-2r$ black points on branch b_j, listed moving outward from point a, map in a way that maintains order to the corresponding $s-2r$ points on branch b_{j+1}.

(2) Since P is unimodal, to finalize the orbit of P, the final $3r-s$ red points located on branch b_j, taken in order starting from the branch point a, must be mapped to the initial $3r-s$ points on branch b_{j+2}, also measured outward from a.

Furthermore, the r black points on branch b_{j+2} are mapped onto branch b_j, also preserving order.

(3) Thus, the $s-2r$ black points $y_1, y_2, \ldots, y_{s-2r}$ in branch b_{j+1} must map to the final $s-2r$ points $z_{3r-s+1}, z_{3r-s+2}, \ldots, z_r$ in branch b_{j+2}, preserving the order and moving outward from point a.

Depending on which branch b_j, with $j = 0, 1, 2$, contains the block C, we obtain three distinct unimodal triod twist patterns denoted by $\Gamma^j_{\frac{r}{s}}$, for $j = 1, 2, 3$. All these patterns possess the same rotation number $\frac{r}{s}$.

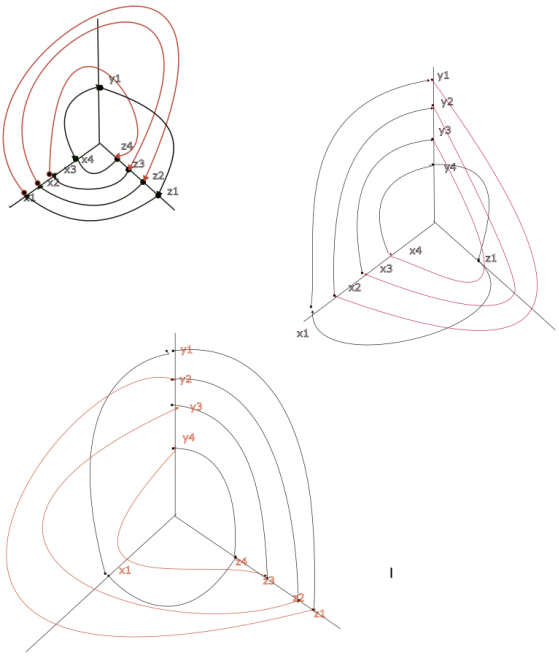


Figure 6.4: $\Gamma^i_{4/9}$ for $i = 1,2,3$ corresponding to $\rho = \frac{4}{9} > \frac{1}{3}$.

6.6 Conclusion

For any positive rational number $\frac{r}{s}$ strictly less than $\frac{1}{2}$ and not equal to $\frac{1}{3}$, there exist exactly three distinct triod-twist patterns, which has one critical point with rotation number $\frac{r}{s}$.

If the rotation number is $\frac{1}{3}$, then there exists a unique unimodal triod twist pattern.

Furthermore, there are no unimodal triod-twist patterns whose rotation number is at least $\frac{1}{2}$.

Chapter 7

Characterization of Bimodal Triod-Twist Patterns

Let us suppose that the branches b_i, for $i = 0, 1, 2$, are ordered in a standard way such that the black block B_i is the one located nearest to the fixed point a on each branch. Given that P is bimodal, there can be at most two additional color blocks, C and D, with C positioned above B_j and D above B_{j+1}, for certain index $j \in \{1, 2, 3\}$.

7.1 For the Case of Rotation Number$< \frac{1}{3}$

We now first focus on the situation where $\frac{p}{q} < \frac{1}{3}$.By lemma[7.3.1] , C, D are green. Let $|C| = g_1, |Bj| = b_1, |D| = g_2, |Bj+1| = b_2$ and $|Bj+3| = b_3$.Then, simple calcution yields us $b_1 + b_2 + b_3 = 3p$ and $g_1 + g_2 + b_1 + b_2 + b_3 = q$. Clearly, $b_1 = b_2 = b_3 = p$ and $g_1 + g_2 = q - 3p$. Let $g_1 = r$, then $g_2 = q - 3p - r$. Clearly, $r \geq 1$, If $r = 0$ our bimodal twist pattern simplifies to a unimodal twist pattern. Consequently, the branches b_j, b_{j+1}, and b_{j+2} (with indices interpreted modulo 3) contain $p + r$, $q - 2p - r$, and p points, respectively. We label these points as $x_1, x_2, \ldots, x_{p+r}$, $y_1, y_2, \ldots, y_{q-2p-r}$, and $z_1, z_2, \ldots, z_p$,in each respective branch, arranged outward from the branching point a

(1) The first p points on branch b_j, ordered outward from point a, are mapped in an order-preserving way to the last p points on branch b_{j+1}, also counted outward from a.

(2) Consequently, the remaining points $y_1, y_2, \ldots, y_{q-3p-r}$ on branch b_{j+1} must be the images of the points $y_p, y_{p+1}, \ldots, y_{q-2p-r}$ from the same branch,which are green.

(3) The first p points of the branch b_{j+1} (in the direction away from a) map to the p points in the branch b_{j+2} in an order preserving fashion.

(4) The final point x_{r+p} in branch b_j (measured away from a) must come from a point in branch b_{j+2} (with indices taken modulo 3). Since P is bimodal, the mapping from branch b_{j+2} to branch b_j does not allow the spacing between points to increase.

(5) Now, the remaining points $x_1, x_2, \ldots, x_r$ in branch b_j must be the images of the points $x_p, x_{p+1}, \ldots, x_r$, respectively, within the same branch.

7.2 For the Case of Rotation Number$> \frac{1}{3}$

We now focus on the situation where $\frac{p}{q} > \frac{1}{3}$. In this case, C and D are red. If the cardinalities of the sets are given by $|C| = r_1$, $|D| = r_2$, $|B_j| = b_1$, $|B_{j+1}| = b_2$, and $|B_{j+2}| = b_3$, then by straightforward calculations we obtain the following relationships:

$$b_2 = q - 2p, \quad r_1 + r_2 = 3p - q, \quad b_1 = q - 2p + r_2, \quad \text{and} \quad b_3 = q - 2p + r_1.$$

Notice that if $\frac{p}{q} \geq \frac{1}{2}$, then $b_2 \leq 0$, which is impossible, meaning no bimodal triod twist cycle exists for this case. Let $r_1 = r$, then $r_2 = 3p - q - r$. To describe the bimodal, we impose the restriction that $r \geq 1$, otherwise it simplifies to unimodal twist pattern. Hence, $r \in 1, 2, 3, ..., 3p - q - 1$.

We label the points of P along the branches b_j, b_{j+1}, and b_{j+2} (moving away from the branching point a) in the same manner as before: $x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_{p-r}$, and $z_1, z_2, \ldots, z_{q-2p+r}$.

(1) Begin with the first $p - r$ black points situated along branch b_j, starting from the branching point a. These points are mapped, in the same order, to the last $p - r$ points on branch b_{j+1}, counted outward from a.

(2) The collection of $q - 2p + r$ black points positioned on branch b_{j+2} are each sent to the terminal $q - 2p + r$ locations on branch b_j, preserving their original order from the center point a.

(3) Moving to the red points: the final $3p - r - q$ such points along branch b_{j+1}, measured outward from a, are mapped in sequence to the first $3p - r - q$ points along branch b_j, also in order from the branching point.

(4) Since P is a bimodal cycle, we require that the last r red points on branch b_j (again, measured from a) be sent to the initial r points on branch b_{j+2}. Specifically, for each $i = 1, 2, \ldots, r$, we have $f(x_{p-r+i}) = z_i$.

(5) Finally, consider the $q-2p$ black points labeled $y_1, y_2, \ldots, y_{q-2p}$ on branch b_{j+1}. These must be mapped, preserving their order from a, to the last $q-2p$ points on branch b_{j+2}, namely $z_{r+1}, z_{r+2}, \ldots, z_{r+q-2p}$.

7.3 Conclusion

For any positive rational number $\frac{r}{s}$ strictly less than $\frac{1}{2}$ and not equal to $\frac{1}{3}$, there exist distinct bimodal triod twist patterns with rotation number $\frac{r}{s}$.

When the rotation number equals $\frac{1}{3}$, there is exactly one bimodal triod twist pattern.

Furthermore, bimodal triod twist patterns do not exist for rotation numbers greater than or equal to $\frac{1}{2}$.

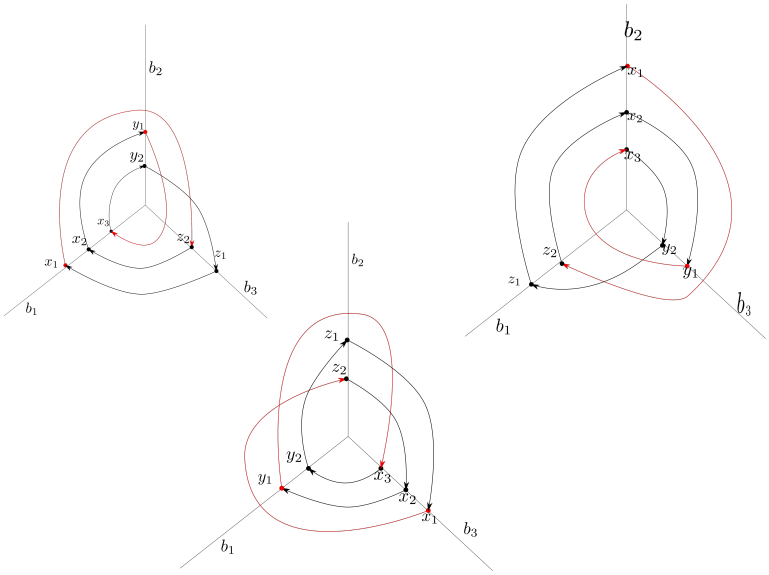


Figure 7.2: The bimodal triod twist patterns $\Gamma_i^{3/7}$ for $i = 1,2,3$ correspond to $\rho = \frac{3}{7} > \frac{1}{3}$.

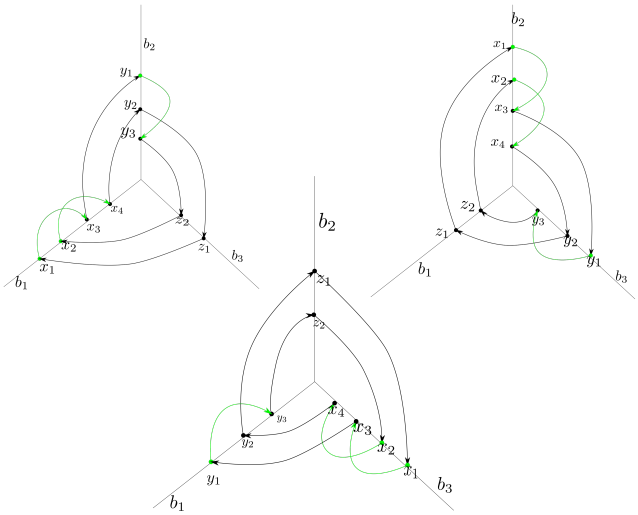


Figure 7.1: The bimodal triod twist patterns $\Gamma_{2/9}^i$ for $i = 1,2,3$ corresponding to $\rho = \frac{2}{9} < \frac{1}{3}$.

Chapter 8

Conclusion and Future Work

Starting from Sharkovsky's theorem and the study of periodicity, we explored rotation theory for both circle maps and interval maps—areas that have been well-studied in the literature. Building on the work in [3], we developed a deeper understanding of unimodal triod-twist patterns and extended this framework to define and analyze bimodal triod-twist patterns ourselves.

For future work, we aim to develop a generalized framework for describing n-ods for higher values of n. This would allow us to further investigate and better understand the increasing complexity that arises in higher-order n-ods.

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